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On a Conjecture of C. A. Micchelli concerning Cubic Spline Interpolation at a Biinfinite Knot Sequence

RONG-QING JIA

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706, U.S.A.

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It is shown that if the knot sequence $\mathbf{t} := (t_i)_{-\infty}^{\infty}$ satisfies (i) for some $m \in [1, (3+\sqrt{5})/2)$,

$$m^{-1} \leqslant \varliminf_{r} \inf_{i} \frac{t_{i+r-1} - t_{i+r}}{t_{i+1} - t_i} \leqslant \varlimsup_{r} \sup_{i} \frac{t_{i+r+1} - t_{i+r}}{t_{i+1} - t_i} \leqslant m$$

and (ii)

$$m_{\mathbf{t}} := \sup_{|i-j| \leqslant 1} \frac{t_{i+1} - t_i}{t_{j+1} - t_j} < \infty,$$

then for any given bounded sequence $y \in m(\mathbb{Z})$ there exists exactly one bounded cubic spline s with knots t_i such that $s(t_i) = y_i$ for all $i \in \mathbb{Z}$.

1. INTRODUCTION

Let $\mathbf{t} := (t_i)_{-\infty}^{\infty}$ be a biinfinite strictly increasing sequence, set

$$t_{\pm\infty} := \lim_{i \to \pm\infty} t_i,$$

let k be an integer, $k \geqslant 2$, and denote by $\mathbb{S}_{k,\mathbf{t}}$ the collection of spline functions of degree $<k$ with knot sequence $\mathbf{t}$. Explicitly, $\mathbb{S}_{k,\mathbf{t}}$ consists of exactly those $(k-2)$-times continuously differentiable functions on $I := (t_{-\infty}, t_{\infty})$ which, on each interval (t_i, t_{i+1}), coincide with some polynomial of degree $<k$. Let

$$m\mathbb{S}_{k,\mathbf{t}} := \mathbb{S}_{k,\mathbf{t}} \cap m(I),$$

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i.e., the normed linear space of splines for which

$$\|s\|_\infty := \sup_{t \in I} |s(t)|$$

is finite. We are interested in the

Bounded Interpolation Problem (B.I.P.). To construct, for given $y \in m(\mathbb{Z})$, some $s \in m\mathbb{S}_{k,\mathbf{t}}$ for which

$$s|_{\mathbf{t}} = y. \tag{1}$$

We will say that the B.I.P. is correct (for the given knot sequence $\mathbf{t}$) if it has exactly one solution for each $y \in m(\mathbb{Z})$.

In case $k = 4$ (cubic spline interpolation), de Boor [2] showed that if the local mesh ratio

$$m_{\mathbf{t}} := \sup_{|i-j| \leqslant 1} (\Delta t_i/\Delta t_j) \qquad (\text{with } \Delta t_i := t_{i+1} - t_i)$$

is less than $(3 + \sqrt{5})/2$, then the B.I.P. is correct. A similar result was also obtained independently by Zmatrakov [7]. The basic idea of [2] was the exponential decay law, which could be traced back to [1]. This idea was developed in [3, 6]. For cubic spline interpolation, Micchelli further raised the following conjecture (see [6, p. 236]).

Conjecture. If

$$m^{-1} \leqslant \varliminf_r \inf_i (\Delta t_{i+r}/\Delta t_{i-1})^{1/r} \leqslant \varlimsup_r \sup_i (\Delta t_{i+r}/\Delta t_{i-1})^{1/r} \leqslant m \tag{2}$$

for some $m \in [1, (3 + \sqrt{5})/2)$, then the B.I.P. is correct for $k = 4$.

The purpose of this paper is to verify this conjecture in its complete form. This means that, in addition to (2), we also require the local mesh ratio to be finite. This meets the spirit if not the actual statement of the conjecture in [6], since any appliction of [6, Theorem 4] requires that $m_{\mathbf{t}}$ be finite. Thus we will prove

THEOREM 1. *If a knot sequence* $\mathbf{t}$ *satisfies* (2) *with* $1 \leqslant m < (3 + \sqrt{5})/2$ *and*

$$m_{\mathbf{t}} < \infty, \tag{3}$$

then the B.I.P. is correct for $k = 4$.

Remark. This theorem covers de Boor's results for cubic splines with bounded global mesh ratio or with local mesh ratio $<(3 + \sqrt{5})/2$ (see [2, 3]).

2. The Basic Formulae for Cubic Spline Interpolation

For a given knot sequence $\mathbf{t}$, let

$$h_i := t_i - t_{i-1}, \qquad \lambda_i = h_{i+1}/(h_i + h_{i+1}), \\ \mu_i = h_i/(h_i + h_{i+1}), \qquad i \in \mathbb{Z}. \tag{4}$$

If $s \in m\mathbb{S}_{4,\mathbf{t}}$ satisfies (1) for some $y \in m(\mathbb{Z})$, then

$$\lambda_i s'(t_{i-1}) + 2s'(t_i) + \mu_i s'(t_{i+1}) \\ = 3\lambda_i \frac{y_i - y_{i-1}}{h_i} + 3\mu_i \frac{y_{i+1} - y_i}{h_{i+1}}, \qquad i \in \mathbb{Z}. \tag{5}$$

Moreover, it is easy to check that

$$s(x) = \frac{(x - t_{i-1})(t_i - x)^2}{h_i^2} s'(t_{i-1}) - \frac{(x - t_{i-1})^2(t_i - x)}{h_i^2} s'(t_i) \\ + \frac{y_{i-1} + y_i}{2} + \frac{1}{h_i^2}\left(x - \frac{t_{i-1} + t_i}{2}\right) \\ \times [(t_i - x)^2 + 4(t_i - x)(x - t_{i-1}) + (x - t_{i-1})^2] \\ \times \frac{y_i - y_{i-1}}{h_i} \qquad \text{for} \quad x \in [t_{i-1}, t_i]. \tag{6}$$

Let

$$\begin{aligned} A(i,j) &:= 2, && \text{for} \quad j = i, \\ &:= \lambda_i, && \text{for} \quad j = i-1, \\ &:= \mu_i, && \text{for} \quad j = i+1, \\ &:= 0, && \text{for} \quad j \in \mathbb{Z}\backslash\{i-1, i, i+1\}, \quad i, j \in \mathbb{Z}. \end{aligned} \tag{7}$$

Then A is a tri-diagonal $\mathbb{Z} \times \mathbb{Z}$-matrix. For any $\beta \in m(\mathbb{Z})$ we have

$$|\lambda_i \beta_{i-1} + 2\beta_i + \mu_i \beta_{i+1}| \leqslant \lambda_i \|\beta\|_\infty + 2\|\beta\|_\infty + \mu_i \|\beta\|_\infty = 3\|\beta\|_\infty, \qquad i \in \mathbb{Z},$$

showing $\|A\| \leqslant 3$. Here, we view A as a mapping from $m(\mathbb{Z})$ to $m(\mathbb{Z})$. Furthermore,

$$|\lambda_i \beta_{i-1} + 2\beta_i + \mu_i \beta_{i+1}| \geqslant 2|\beta_i| - \lambda_i \|\beta\| - \mu_i \|\beta\| \\ = 2|\beta_i| - \|\beta\|_\infty, \qquad i \in \mathbb{Z}.$$

Hence $\|A\beta\|_\infty \geqslant \sup_{i\in\mathbb{Z}}\{2|\beta_i| - \|\beta\|_\infty\} = \|\beta\|_\infty$. This shows that A^{-1} exists and $\|A^{-1}\| \leqslant 1$.

3. The Exponential Decay

The following lemma plays an essential role in this paper:

Lemma 1. *For any knot sequence* **t**,

$$|A^{-1}(j,i)| \leqslant 2^{-|j-i|}, \qquad i,j\in\mathbb{Z}, \tag{8}$$

and

$$A^{-1}(j,i)\,A^{-1}(j+1,i) < 0, \qquad i,j\in\mathbb{Z}. \tag{9}$$

Moreover, if **t** *satisfies the following condition: For some integer* $r \geqslant 0$ *and real number* $m_0 > 0$

$$h_i/h_j \leqslant \tfrac{3}{4} m_0^{|i-j|} \qquad \text{whenever} \quad |i-j| \geqslant r, \tag{10}$$

then

$$|A^{-1}(j,i)| \leqslant (1 + m_0^{-1} + \sqrt{1 + m_0^{-1} + m_0^{-2}})^{-|i-j|} \qquad \text{whenever} \quad |i-j| \geqslant r. \tag{11}$$

Proof. For simplicity we fix i and write $b_j := A^{-1}(j,i)$. Since $\|A^{-1}\| \leqslant 3$, $b\in m(\mathbb{Z})$. By $AA^{-1} = 1$ we have

$$\lambda_j b_{j-1} + 2b_j + \mu_j b_{j+1} = \delta_{ij} := 1, \quad \text{for} \quad j = i, \qquad := 0, \quad \text{for} \quad j \neq i. \tag{12}$$

We claim that

$$|b_j| \leqslant |b_{j+1}| \qquad \text{for all} \quad j < i. \tag{13}$$

If not, then there exists $j_0 < i$ such that $|b_{j_0}| > |b_{j_0+1}|$. From (12) we have

$$\begin{aligned} |b_{j_0-1}| &= |-(2b_{j_0} + \mu_{j_0} b_{j_0+1})/\lambda_{j_0}| \geqslant (2|b_{j_0}| - \mu_{j_0}|b_{j_0+1}|)/\lambda_{j_0} \\ &\geqslant (2-\mu_{j_0})|b_{j_0}|/\lambda_{j_0} = |b_{j_0}|(1+\lambda_{j_0})/\lambda_{j_0} \geqslant 2|b_{j_0}|. \end{aligned}$$

Then by induction on j, we can easily show that $|b_{j-1}| \geqslant 2|b_j|$ for all $j \leqslant j_0$. Hence $|b_j| \geqslant 2^{j_0-j}|b_{j_0}|$ for $j < j_0$. This contradicts the fact $b\in m(\mathbb{Z})$. One proves similarly that

$$|b_j| \leqslant |b_{j-1}| \qquad \text{for all} \quad j > i. \tag{13'}$$

Now (12) and (13) yield that

$$|b_{j+1}| = |2b_j + \lambda_j b_{j-1}|/\mu_j \geqslant (2|b_j| - \lambda_j |b_{j-1}|)/\mu_j \qquad (14)$$
$$\geqslant (2-\lambda_j)|b_j|/\mu_j = |b_j| \cdot (1+\mu_j)/\mu_j \geqslant 2|b_j| \qquad \text{for} \quad j < i.$$

Similarly

$$|b_{j-1}| \geqslant 2|b_j| \qquad \text{for} \quad j > i. \qquad (14')$$

In particular, $|b_{i-1}| \leqslant |b_i|$ and $|b_{i+1}| \leqslant |b_i|$. In connection with (12) we obtain

$$1 = \lambda_i b_{i-1} + 2b_i + \mu_i b_{i+1} = |\lambda_i b_{i-1} + 2b_i + \mu_i b_{i+1}|$$
$$\geqslant 2|b_i| - \lambda_i |b_{i-1}| - \mu_i |b_{i+1}| \geqslant (2 - \lambda_i - \mu_i)|b_i| = |b_i|. \qquad (15)$$

This proves (8) for $j = i$. For $j \neq i$, (8) comes from (14), (14′), and (15). For the rest of the proof we may assume $j < i$ without any loss of generality. To prove (9) we argue indirectly. If $b_{j_0} b_{j_0+1} \geqslant 0$ for some $j_0 < i$, then

$$|b_{j_0-1}| \geqslant |\lambda_{j_0} b_{j_0-1}| = |2b_{j_0} + \mu_{j_0} b_{j_0+1}| \geqslant 2|b_{j_0}|.$$

Comparing the above inequality with (13), we must have $b_j = 0$ for all $j \leqslant j_0$. This would cause all $b_j = 0$, which is absurd. Now we can write

$$-b_{j+1}/b_j =: 2 + 2q_j' \qquad \text{for} \quad j < i \qquad (16)$$

with $q_j' \geqslant 0$. Let $q_j := h_{j+1}/h_j$. We deduce from (12) that, for $j < i-1$,

$$-\frac{b_{j+2}}{b_{j+1}} = \frac{2b_{j+1} + \lambda_{j+1} b_j}{\mu_{j+1} b_{j+1}} = \frac{2}{\mu_{j+1}} + \frac{\lambda_{j+1}}{\mu_{j+1}} \frac{b_j}{b_{j+1}}$$
$$= 2 + 2q_{j+1} + q_{j+1} \frac{-1}{2 + 2q_j'}$$
$$= 2 + 2q_{j+1}\left(1 - \frac{1}{2(2+2q_j')}\right) = 2 + 2q_{j+1} \frac{4q_j' + 3}{4q_j' + 4}.$$

This shows that

$$q_{j+1}' = q_{j+1} \frac{4q_j' + 3}{4q_j' + 4} \qquad \text{for} \quad j < i-1. \qquad (17)$$

Let

$$p_j := \frac{4q_j' + 4}{4q_j' + 3} q_j'. \qquad (18)$$

It is easy to verify that

$$2 + 2q_j' = 1 + p_j + \sqrt{1 + p_j + p_j^2}. \tag{19}$$

Now (16) and (19) give us

$$\begin{aligned} |b_i/b_j| &= \prod_{k=j}^{i-1} |b_{k+1}/b_k| = \prod_{k=j}^{i-1} (2 + 2q_k') \\ &= \prod_{k=j}^{i-1} (1 + p_k + \sqrt{1 + p_k + p_k^2}). \end{aligned} \tag{20}$$

It follows from (17) and (18) that

$$\begin{aligned} \prod_{k=j}^{i-1} p_k &= \prod_{k=j}^{i-1} \left(\frac{4q_k' + 4}{4q_k' + 3} q_k'\right) = \prod_{k=j}^{i-1} \left(\frac{4q_k' + 4}{4q_k' + 3} \frac{4q_{k-1}' + 3}{4q_{k-1}' + 4} q_k\right) \\ &= \frac{4q_{i-1}' + 4}{4q_{i-1}' + 3} \frac{4q_{j-1}' + 3}{4q_{j-1}' + 4} \prod_{k=j}^{i-1} q_k \geqslant \frac{3}{4} \prod_{k=j}^{i-1} q_k = \frac{3}{4} \frac{h_i}{h_j}. \end{aligned} \tag{21}$$

If $\mathbf{t}$ satisfies (8) and $|i-j| \geqslant r$, then

$$\prod_{k=j}^{i-1} p_k \geqslant \frac{3}{4} \frac{h_i}{h_j} \geqslant \frac{3}{4} \frac{4}{3} m_0^{-|i-j|} = m_0^{-|i-j|}. \tag{22}$$

Therefore Lemma 1 will be proved, once the following lemma is established:

LEMMA 2. *Suppose $p_1, \ldots, p_n$ and p are nonnegative real numbers with $p^n = p_1 p_2 \cdots p_n$. Then*

$$\prod_{i=1}^{n} (1 + p_i + \sqrt{1 + p_i + p_i^2}) \geqslant (1 + p + \sqrt{1 + p + p^2})^n. \tag{23}$$

Proof. Let $F(x) := \log(1 + x + \sqrt{1 + x + x^2})$, $x := e^t$, and $G(t) = F(e^t)$. Then

$$\begin{aligned} G'(t) &= \frac{dF}{dx} \frac{dx}{dt} = \frac{(1 + ((2x+1)/2\sqrt{1 + x + x^2}))x}{1 + x + \sqrt{1 + x + x^2}} \\ &= \frac{1}{2} + \frac{x - 1}{2\sqrt{1 + x + x^2}}; \end{aligned}$$

$$\begin{aligned} G''(t) &= \frac{d}{dx} \left(\frac{1}{2} + \frac{x - 1}{2\sqrt{1 + x + x^2}}\right) \frac{dx}{dt} \\ &= \frac{3(x+1)x}{4(1 + x + x^2)^{3/2}} \geqslant 0 \qquad \text{for} \quad x \geqslant 0. \end{aligned}$$

This shows that $G(t)$ is a convex function of t. Hence

$$G\left(\frac{1}{n}\sum_{i=1}^{n}\log p_i\right)\leqslant\frac{1}{n}\sum_{i=1}^{n}G(\log p_i), \tag{24}$$

and (23) follows from this by exponentiation. This ends the proof of Lemma 2. Also the proof of Lemma 1 is complete.

4. The Proof of Theorem 1

By hypothesis (2) there exist a positive integer r and a real number m_0 with $m < m_0 < (3+\sqrt{5})/2$ such that

$$h_i/h_j \leqslant \tfrac{3}{4} m_0^{|i-j|} \qquad \text{whenever} \quad |i-j| \geqslant r.$$

Then by Lemma 1, for $i, j \in \mathbb{Z}$,

$$\begin{aligned}|A^{-1}(j,i)| &\leqslant 2^{-|i-j|}, && \text{if} \quad |i-j|<r,\\ &\leqslant (1+m_0^{-1}+\sqrt{1+m_0^{-1}+m_0^{-2}})^{-|i-j|}, && \text{if} \quad |i-j|\geqslant r.\end{aligned} \tag{25}$$

Let

$$M := (m_1)^r < \infty$$

and

$$c_i := 3\lambda_i \frac{y_i - y_{i-1}}{h_i} + 3\mu_i \frac{y_{i+1}-y_i}{h_{i+1}}, \qquad i \in \mathbb{Z}.$$

Then it follows that

$$s'(t_j) = \sum_{i\in\mathbb{Z}} A^{-1}(j,i)\, c_i = \sum_{|i-j|<r} A^{-1}(j,i)\, c_i + \sum_{|i-j|\geqslant r} A^{-1}(j,i)\, c_i. \tag{26}$$

By the hypotheses of Theorem 1 we have the following estimates for c_i:

$$\begin{aligned}|c_i| &\leqslant 6\,\|y\|_\infty (1+M)\frac{1}{h_i} \leqslant 6M(1+M)\,\|y\|_\infty \frac{1}{h_j} && \text{if} \quad |j-i|<r;\\ |c_i| &\leqslant 6\,\|y\|_\infty (1+M)\frac{1}{h_i} \leqslant 6(1+M)\,\|y\|_\infty \frac{1}{h_j} m_0^{|i-j|} && \text{if} \quad |j-i|\geqslant r.\end{aligned} \tag{27}$$

Write

$$\theta := m_0(1+m_0^{-1}+\sqrt{1+m_0^{-1}+m_0^{-2}})^{-1}.$$

Then $\theta < 1$ as long as $m_0 < (3+\sqrt{5})/2$. Applying (25) and (27) to (26), we obtain

$$|s'(t_j)| \leqslant 6M(1+M)\,\|y\|_\infty \frac{1}{h_j} \sum_{|j-i|<r} 2^{-|i-j|} + 6(1+M)\,\|y\|_\infty \frac{1}{h_j} \sum_{|i-j|\geqslant r} \theta^{|i-j|} \leqslant \text{const}\,\|y\|_\infty \frac{1}{h_j}. \tag{28}$$

Furthermore (6) tells us that

$$\max_{t_{j-1}\leqslant x\leqslant t_j} |s(x)| \leqslant \text{const}(h_{j-1}\,|s'(t_{j-1})| + h_j\,|s'(t_j)| + \|y\|_\infty),$$

which in connection with (28) yields the desired result

$$\|s\|_\infty \leqslant \text{const}\,\|y\|_\infty.$$

Our proof is complete.

Remark. Professor Carl de Boor once suggested to me that Lemma 1 could be generalized. Also, the referee guessed that each example in [6] could be improved similarly. Indeed, one can get a slightly more general form of Lemma 1 and apply it to spline interpolation. In particular, the result in [6, p. 234, Example 4] can be improved. However, our method yields no improvement for Examples 2 and 3. This is understandable because the matrices in Examples 2 and 3 are not always diagonally dominant. We shall discuss this matter in detail elsewhere.

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